Optimal policy in Rational Expectations Model: New Solution Algorithms

Dennis (2005 *forthcoming*) Macro Dynamics

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Motivation

- Previous solution algorithms Oudiz and Sachs (1985), Currie and Levine (1985,1993), Backus and Driffill (1986), and Soderlind (1999) require constraints to be written in state-space form
- Allow constraints in structural form, no need to distinguish between predetermine and jump variables
- Include next period’s policy instrument(s) in the optimization constraint
- Easier to apply in practice especially when one is considering many different medium-scale models

Overview of the paper

- ✓ Small contribution but significant in terms of practicality
- ✓ “New” algorithm to solve optimal policy in rational expectation models
- ✓ Central bank’s control problem, under both commitment and discretion
- ✓ Easily understood
- ✓ Applied to three different models, test for efficiency
- ✓ Provide some insights into the convergence problems

Compare to S.S specification

- Soderlind (1999) and co

$$Loss(t,\infty) = E_t \sum_{t=0}^{\infty} \beta^t [\mathrm{y}_t' W \mathrm{y}_t + \mathrm{x}_t' Q \mathrm{x}_t], \tag{1}$$

$$\text{s.t:} \quad \mathrm{y}_{t+1} = A\mathrm{y}_t + B\mathrm{x}_t + C\varepsilon_{t+1}, \tag{2}$$

$$\text{where} \quad \mathrm{y}_{t+1} = \begin{bmatrix} \mathrm{y}_{1t+1} & E_t \mathrm{y}_{2t+1} \end{bmatrix}'.$$

- Dennis (2005)

$$Loss(t,\infty) = E_t \sum_{t=0}^{\infty} \beta^t [\mathrm{y}_t' W \mathrm{y}_t + \mathrm{x}_t' Q \mathrm{x}_t], \tag{3}$$

$$\text{s.t:} \quad A_0 \mathrm{y}_t = A_1 \mathrm{y}_{t-1} + A_2 E_t \mathrm{y}_{t+1} + A_3 \mathrm{x}_t + A_4 E_t \mathrm{x}_{t+1} + A_5 v_t. \tag{4}$$

Commitment - The Problem

CB choose the path of y_t, x_t and λ_t, $\forall t \geq 0$, and commit to it:

$$
\begin{aligned}
L = E_t \sum_{t=0}^{\infty} \beta^t [\mathrm{y}_t' W \mathrm{y}_t + \mathrm{x}_t' Q \mathrm{x}_t + 2\lambda_t' (A_0 \mathrm{y}_t - A_1 \mathrm{y}_{t-1} \\
- A_2 \mathrm{y}_{t+1} - A_3 \mathrm{x}_t - A_4 \mathrm{x}_{t+1} - \rho_t)]
\end{aligned}
\tag{5}
$$

$$
\frac{dL}{d\mathrm{x}_t} = Q\mathrm{x}_t - A_3'\lambda_t - \beta^{-1} A_4' \lambda_{t-1} = 0, \;\; t > 0; \tag{6a}
$$

$$
\frac{dL}{d\mathrm{y}_t} = W\mathrm{y}_t + A_0'\lambda_t - \beta^{-1} A_2' \lambda_{t-1} - \beta A_1' E_t \lambda_{t+1} = 0, \;\; t > 0; \tag{6b}
$$

$$
\frac{dL}{d\lambda_t} = A_0 \mathrm{y}_t - A_1 \mathrm{y}_{t-1} - A_2 E_t \mathrm{y}_{t+1} - A_3 \mathrm{x}_t - A_4 E_t \mathrm{x}_{t+1} - A_5 v_t = 0, \; t \geq 0; \tag{6c}
$$

$$
\frac{dL}{d\mathrm{x}_t} = Q\mathrm{x}_t - A_3'\lambda_t = 0, \;\; t = 0; \tag{6d}
$$

$$
\frac{dL}{d\mathrm{y}_t} = W\mathrm{y}_t + A_0'\lambda_t - \beta A_1' E_t \lambda_{t+1} = 0, \;\; t = 0. \tag{6e}
$$

Only (6a) to (6c) will be binding, or $\lambda_0 = 0$

Commitment - The solution

Write equations (6a-6c) as $A\mathrm{Z}_t = B\mathrm{Z}_{t-1} + CE_t\mathrm{Z}_{t+1} + Dv_t$:

$$\underbrace{\begin{bmatrix} 0 & A_0 & -A_3 \\ A_0' & W & 0 \\ -A_3' & 0 & Q \end{bmatrix}}_{(2n+p)\times(2n+p)} \begin{bmatrix} \lambda_t \\ \mathrm{y}_t \\ \mathrm{x}_t \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & A_1 & 0 \\ \beta_{-1}A_2' & 0 & 0 \\ \beta_{-1}A_4' & 0 & 0 \end{bmatrix}}_{(2n+p)\times(2n+p)} \begin{bmatrix} \lambda_{t-1} \\ \mathrm{y}_{t-1} \\ \mathrm{x}_{t-1} \end{bmatrix} \qquad (7)$$

$$+ \underbrace{\begin{bmatrix} 0 & A_2 & A_4 \\ \beta A_1' & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{(2n+p)\times(2n+p)} E_t \begin{bmatrix} \lambda_{t+1} \\ \mathrm{y}_{t+1} \\ \mathrm{x}_{t+1} \end{bmatrix} + \underbrace{\begin{bmatrix} A_5 \\ 0 \\ 0 \end{bmatrix}}_{(2n+p)\times s} [v_t]$$

Binder and Pesaran (1995) Solution: $\mathrm{Z}_t = H\mathrm{Z}_{t-1} + Gv_t \qquad (8)$

Commitment - Remarks

1. Can also solve equations (6a-6c) using eigenvalue methods
2. The Euler equations (6a-6c) do not depend on Ω - certainty equivalent. Special property under quadratic objective, linear constraints and v_t is a Martingale sequence
3. $Loss(t,\infty) = \left[\mathrm{Z}_t' \hat{P} \mathrm{Z}_t + \frac{\beta}{1-\beta} tr(G'\hat{P}G\Omega) \right]$, where

   $\hat{P} \equiv \hat{K} + \beta H'\hat{P}H$, and $\hat{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & Q \end{bmatrix}$
4. $\lim_{\beta \uparrow 1} (1-\beta) Loss(t,\infty) = tr(K\Phi)$, where Φ is the unconditional var-cov of $[\mathrm{y}_t' \;\; \mathrm{x}_t']'$ and $K = \begin{bmatrix} W & 0 \\ 0 & Q \end{bmatrix}$

Discretion - The Problem

$$Loss(t,\infty) = E_t \sum_{t=0}^{\infty} \beta^t [\mathrm{y}_t' W \mathrm{y}_t + \mathrm{x}_t' Q \mathrm{x}_t], \tag{9}$$

$$\text{s.t:} \quad A_0 \mathrm{y}_t = A_1 \mathrm{y}_{t-1} + A_2 E_t \mathrm{y}_{t+1} + A_3 \mathrm{x}_t + A_4 E_t \mathrm{x}_{t+1} + A_5 v_t. \tag{10}$$

Conjecture:

$$\mathrm{y}_t = H_1 \mathrm{y}_{t-1} + H_2 v_t \tag{11}$$

$$\mathrm{x}_t = F_1 \mathrm{y}_{t-1} + F_2 v_t \tag{12}$$

$$Loss(t,\infty) = \mathrm{y}_t' P \mathrm{y}_t' + \mathrm{x}_t' Q \mathrm{x}_t' + \frac{\beta}{1-\beta} tr[(F_2' Q F_2 + H_2' P H_2)\Omega] \tag{13}$$

$$\text{s.t:} \quad D \mathrm{y}_t = A_1 \mathrm{y}_{t-1} + A_3 \mathrm{x}_t + A_5 v_t. \tag{14}$$

$$P \equiv W + \beta F_1' Q F_1 + H_1' P H_1 \ \text{ and } \ D \equiv A_0 - A_2 H_1 - A_4 F_4$$

Discretion - The solution

$$
\begin{aligned}
Loss(t,\infty) =&(A_1\mathrm{y}_{t-1}+A_3\mathrm{x}_t+A_5v_t)'D'^{-1}PD^{-1}(A_1\mathrm{y}_{t-1}+A_3\mathrm{x}_t+A_5v_t)\\
&+\mathrm{x}_t'Q\mathrm{x}_t+\frac{\beta}{1-\beta}tr[(F_2'QF_2+H_2'PH_2)\Omega]
\end{aligned}
$$

$$
\begin{aligned}
\frac{dLoss(t,\infty)}{d\mathrm{x}_t} &= A_2'D'^{-1}PD^{-1}(A_1\mathrm{y}_{t-1}+A_3\mathrm{x}_t+A_5v_t)+Q\mathrm{x}_t\\
&= A_2'D'^{-1}PD^{-1}(A_1\mathrm{y}_{t-1}+A_5v_t)+(Q+A_3'D'^{-1}PD^{-1}A_3)\mathrm{x}_t=0
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{x}_t &= (Q+A_3'D'^{-1}PD^{-1}A_3)^{-1}A_2'D'^{-1}PD^{-1}(A_1\mathrm{y}_{t-1}+A_5v_t) && (15a)\\
&\equiv F_1\mathrm{y}_{t-1}+F_2v_t\\
\mathrm{y}_t &= D^{-1}(A_1+A_3F_1)\mathrm{y}_{t-1}+D^{-1}(A_5+A_3F_2)v_t && (15b)\\
&\equiv H_1\mathrm{y}_{t-1}+H_2v_t
\end{aligned}
$$

Given H_1^0, H_2^0, F_1^0 and $F_2^0 \Rightarrow$ solve for H_1, H_2, F_1 and F_2

Solving a Ricatti equation

A simple example: $G = AGB + V.$

S1: Given G_0, start at $j = 1$;

S2: Calculate G_1 using $G_1 = AG_0B + V$;

S3: Repeat S2 using $G_{j+1} = AG_jB + V$;

S4: Stop when $|G_{j+1} - G_j| \leq \epsilon.$

```
function [g1]=ricatti(a, b, v, eps, maxiters)

g0=.01*eye(length(a)); j= 1; lenj = 1;
 while lenj >= eps && j<=maxiters;
    g1 = v+a*g0*b;
    lenj = max(max(abs(g1-g0)));
    g0 = g1;
  j = j + 1;
 end;
```

Discretion - Remarks

1. Solving for H_1, H_2, F_1 and F_2 is generally very efficient
2. The matrices H_1, H_2, F_1 and F_2 are independent of Ω and y_0, certainty equivalent result still holds.
3. $Loss(t,\infty) = \left[\mathrm{Z}_t'\tilde{P}\mathrm{Z}_t + \frac{\beta}{1-\beta}tr(G'\tilde{P}G\Omega)\right]$, where

   $\mathrm{Z}_t = [\mathrm{y}_t' \;\; \mathrm{x}_t']'$, $\tilde{P} \equiv K + \beta H'\tilde{P}H$, and $K = \begin{bmatrix} W & 0 \\ 0 & Q \end{bmatrix}$
4. $\lim_{\beta\uparrow 1}(1-\beta)Loss(t,\infty) = tr(K\Phi)$, where Φ is the unconditional var-cov of Z_t
5. Solution under discretion is generally faster than commitment because of the smaller state vector, $\mathrm{Z}_t = [\mathrm{y}_t' \;\; \mathrm{x}_t']'$ rather than $\mathrm{Z}_t = [\lambda_t' \;\; \mathrm{y}_t' \;\; \mathrm{x}_t']'$

A simple application

- Apply to Gali and Monacelli (2004) to see if the new algorithm is “practical” for estimation purposes
- The model:

$$y_t = E_t y_{t+1} - \frac{1 + \alpha(2-\alpha)(\sigma\eta - 1)}{\sigma}(i_t - E_t\pi_{t+1}) + g_t \tag{16a}$$

$$\pi_t = \beta E_t \pi_{t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\left(\phi + \frac{\sigma}{1 + \alpha(2-\alpha)(\sigma\eta - 1)}\right) y_t + u_t \tag{16b}$$

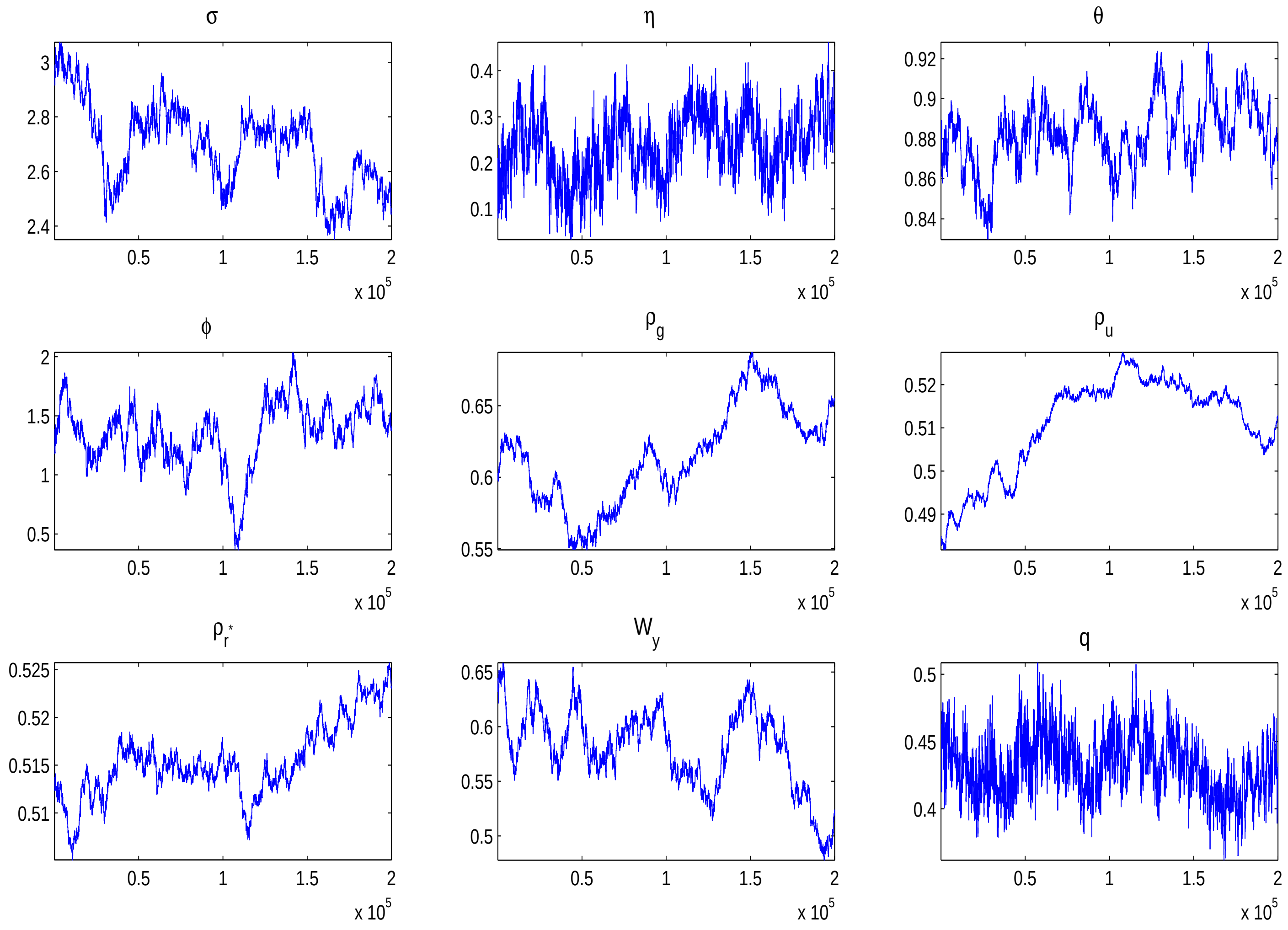
$$\pi_t^c = \pi_t + \frac{\alpha}{1-\alpha}(q_t - q_{t-1}) \tag{16c}$$

$$q_t = E_t q_{t+1} - (1-\alpha)(i_t - E_t\pi_{t+1} - r_t^*) + (1-\alpha)\epsilon_t^q \tag{16d}$$

g_t, u_t and r_t^* are AR(1) processes with white noise terms $\epsilon_t^g, \epsilon_t^u, \epsilon_t^{r^*}$

- Takes around 5 hours to simulate 100k MCMC draws under discretion, and 7.5 hours under commitment on a P4 2.4G
- Haven’t managed to get MLE working yet

Markov Chain - Discretion



Markov Chain - Discretion